Lesson 7 – Logarithms and Logarithmic Functions

Logarithms are exponents. In this Lesson, you will start by working with the LOG button on your calculator and then building an understanding of logarithms as exponents. You will learn how to read and interpret logarithms and how to compute with base 10 and other bases as well.

Prior to solving logarithmic equations, you will learn about changing back and forth form logarithmic to exponential forms. Finally, you will use what you learned about changing forms to solve logarithmic and exponential equations. Pay close attention to the idea of exact form vs. approximate form for solutions.

Lesson Topics:

Section 7.1: Introduction to Logarithms

- Discuss the concept of Logarithms as Exponents
- Compute logarithms with base 10 (Common Logarithm)
- Change an equation from logarithmic form to exponential form and vice versa

Section 7.2: Computing Logarithms

- Compute logarithms with bases other than 10
- Properties of Logarithms
- The Change of Base Formula

Section 7.3: Characteristics of Logarithmic Functions

- Use the Change of Base Formula to graph a logarithmic function and identify important characteristics of the graph.

Section 7.4: Solving Logarithmic Equations

- Solve logarithmic equations algebraically by changing to exponential form.
- Determine EXACT FORM and APPROXIMATE FORM solutions for logarithmic equations

Section 7.5: Solving Exponential Equations Algebraically and Graphically

- Determine EXACT FORM and APPROXIMATE FORM solutions for exponential equations

Section 7.6: Using Logarithms as a Scaling Tool

Lesson 7 Checklist

Component	Required? Y or N	Comments	Due	Score
Mini-Lesson				
Online Homework				
Online Quiz				
Online Test				
Practice Problems				
Lesson Assessment				

Name: ______________________________ **Date:** ________________

Mini-Lesson 7

Section 7.1 – Introduction to Logarithms

Logarithms are really EXPONENTS in disguise. The following two examples will help explain this idea.

Problem 1	YOU TRY – COMPUTE BASE 10 LOGARITHMS USING YOUR CALCULATOR

Locate the **LOG** button on your calculator. Use it to fill in the missing values in the input/output table. The first and last are done for you. When you use your calculator, remember to close parentheses after your input value.

x	$y = \log(x)$
1	0
10	
100	
1000	
10000	
100000	5

What do the outputs from Problem 1 really represent? Where are the EXPONENTS that were mentioned previously? Let's continue with the example and see where we end up.

Problem 2	**MEDIA EXAMPLE – LOGARITHMS AS EXPONENTS**

x	$\log(x)$		$\log_{10}(x) = y$	$10^y = x$
1	0			
10	1			
100	2			
1000	3			
10000	4			
100000	5			

Reading and Interpreting Logarithms

$$\log_b x = y$$

Read this as "Log, to the BASE b, of *x*, equals *y*"

This statement is true if and only if

$$b^y = x$$

Meaning:
The logarithm (output of $\log_b x$) is the EXPONENT on the base, b, that will give you input *x*.

Note: The Problem 2 logarithm is called a COMMON LOGARITHM because the base is understood to be 10. When there is no base value written, you can assume the base = 10.

$$\log(x) = \log_{10}(x)$$

Problem 3	**MEDIA EXAMPLE – EXPONENTIAL AND LOGARITHMIC FORMS**

Complete the table.

	Exponential Form	Logarithmic Form
a)	$\square^{\square} = \square$	$\text{Log}_{\square}\square = \square$
b)	$6^3 = 216$	
c)	$5^{-2} = \frac{1}{25}$	
d)		$\log_7 16807 = 5$
e)		$\log x = 5$

Note: When you write expressions involving logarithms, be sure the base is a SUBSCRIPT and written just under the writing line for Log. Pay close attention to how things are written and what the spacing and exact locations are.

Problem 4	**YOU TRY – EXPONENTIAL AND LOGARITHMIC FORMS**

Complete the table.

	Exponential Form	Logarithmic Form
a)	$3^4 = 81$	
b)		$\log x = 6$
c)		$\log_2\left(\frac{1}{8}\right) = -3$

Section 7.2 – Computing Logarithms

Below are some basic properties of exponents that you will need to know.

Properties of Exponents		
$b^0 = 1$	$\frac{1}{b} = b^{-1}$	$\sqrt{b} = b^{1/2}$
$b^1 = b$	$\frac{1}{b^n} = b^{-n}$	$\sqrt[n]{b} = b^{1/n}$

Problem 5	**MEDIA EXAMPLE – COMPUTE LOGARITHMS WITH BASES OTHER THAN 10**

Compute each of the following logarithms and verify your result with an exponential "because" statement.

a) $\log_2 2^4 =$	because	
b) $\log_2 4 =$	because	
c) $\log_3 27 =$	because	
d) $\log_8 1 =$	because	
e) $\log_5 \sqrt{5} =$	because	
f) $\log_4 4 =$	because	

Properties of Logarithms

$\log_b x = y$	because	$b^y = x$
$\log_b 1 = 0$	because	$b^0 = 1$
$\log_b b = 1$	because	$b^1 = b$
$\log_b b^n = n$	because	$b^n = b^n$
$\log_b 0$ does not exist	because	There is no power of b that will give a result of 0.

Problem 6	**WORKED EXAMPLE - COMPUTE LOGARITHMS**

Compute each of the following logarithms and verify your result with an exponential "because" statement.

a) $\log_3 \frac{1}{9} = \log_3 \frac{1}{3^2} = \log_3 3^{-2}$ so $\log_3 \frac{1}{9} = -2$	because	$3^{-2} = \frac{1}{9}$
b) $\log_6 \frac{1}{36} = \log_6 \frac{1}{6^2} = \log_6 6^{-2}$ so $\log_6 \frac{1}{36} = -2$	because	$6^{-2} = \frac{1}{36}$
b) $\log 1000 = \log_{10} 1000 = \log_{10} 10^3$ so $\log 1000 = 3$ *This is the COMMON LOGARITHM (no base written), so the base=10*	because	$10^3 = 1000$
c) $\log_5 1 = 0$	because	$5^0 = 1$
d) $\log_7 0$ does not exist (D.N.E.)	because	There is no power of 7 that will give a result of 0.

Problem 7	**YOU TRY - COMPUTE LOGARITHMS**

Compute each of the following logarithms and verify your result with an exponential "because" statement.

a) $\log_2 64 =$	because	
b) $\log_3 1 =$	because	
c) $\log \frac{1}{1000} =$	because	
d) $\log 0 =$	because	
e) $\log_8 \sqrt{8} =$	because	

Now that we know something about working with logarithms, let's see how our calculator can help us with more complicated examples.

Problem 8	MEDIA EXAMPLE – INTRODUCING CHANGE OF BASE FORMULA

Let's try to compute $\log_2 19$. To start, let's estimate values for this number. Try to find the two consecutive (one right after the other) whole numbers that $\log_2 19$ lives between.

$$\_\_\_\_\_\_\_\_\_\_\_\_ < \log_2 19 < \_\_\_\_\_\_\_\_\_\_\_\_$$

$$\_\_\_\_\_\_\_\_\_\_\_\_ < \log_2 19 < \_\_\_\_\_\_\_\_\_\_\_\_$$

So, now we have a good estimate for $\log_2 19$ let's see how our calculator can help us find a better approximation for the number.

To compute $\log_2 19$ in your calculator, use the following steps: Log>19)>/Log2)>ENTER and round to three decimals to get:

$$\log_2 19 = \_\_\_\_\_\_\_\_\_\_\_\_$$

Do we believe this is a good approximation for $\log_2 19$? How would we check?

____

So, our estimation for $\log_2 19$ is good and we can say $\log_2 19 =$ __________ with certainty.

How did we do that again? We said that $\log_2 19 = \dfrac{\log(19)}{\log(2)}$. How can we do that for any problem?

Change of Base Formula – Converting with Common Logarithms (base 10)

$$\log_b x = \frac{\log(x)}{\log(b)}$$

Problem 9	**YOU TRY – COMPUTE LOGARITHMS USING CHANGE OF BASE FORMULA**

Use the Change of Base formula given on the previous page, and your calculator, to compute each of the following. The first one is done for you.

Compute	Rewrite using Change of Base	Final Result (3 decimal places)
a) $\log_3 8$	$\frac{\log(8)}{\log(3)}$	1.893
b) $\log_5 41$		
c) $\log_8 12$		
d) $\log_{1.5} 32$		
e) $12.8+\log_3 25$		

Section 7.3 – Characteristics of Logarithmic Functions

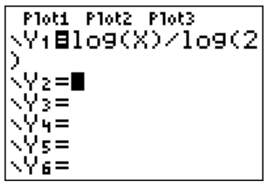
The Change of Base Formula can be used to graph Logarithmic Functions. In the following examples, we will look at the graphs of two Logarithmic Functions and analyze the characteristics of each.

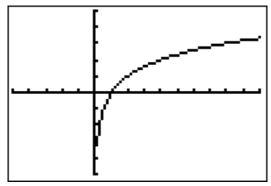
Problem 10	**WORKED EXAMPLE – GRAPHING LOGARITHMIC FUNCTIONS**

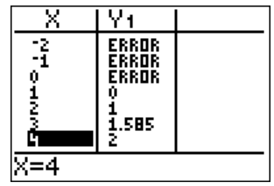
Given the function $f(x) = \log_2 x$, graph the function using your calculator and identify the characteristics listed below. Use window x: [-5..10] and y: [-5..5].

Graphed function: To enter the function into the calculator, we need to rewrite it using the Change of Base Formula, enter that equation into Y_1, and then Graph.

$$f(x) = \log_2 x = \frac{\log x}{\log 2}$$



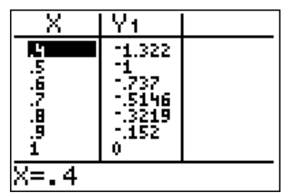




Characteristics of the Logarithmic Functions:

Domain: $x > 0$, Interval Notation: $(0,\infty)$
The graph comes close to, but never crosses the vertical axis. Any input value that is less than or equal to 0 ($x \le 0$) produces an error. Any input value greater than 0 is valid. The table above shows a snapshot of the table from the calculator to help illustrate this point.

Range: All Real Numbers, Interval Notation $(-\infty,\infty)$
The graph has output values from negative infinity to infinity. As the input values get closer and closer to zero, the output values continue to decrease (See the table to the right). As input values get larger, the output values continue to increase. It slows, but it never stops increasing.



Vertical Asymptote at $x = 0$. The graph comes close to, but never crosses the line $x = 0$ (the vertical axis). Recall that, for any base b, $\log_b(0)$ does not exist because there is no power of b that will give a result of 0.

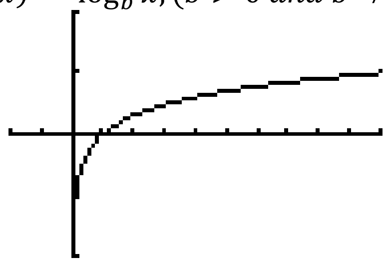
Vertical Intercept: Does Not Exist (DNE).

Horizontal Intercept: (1,0) This can be checked by looking at both the graph and the table above as well as by evaluating $f(1) = \log_2(1) = 0$. Recall that, for any base b, $\log_b(1) = 0$ because $b^0 = 1$.

Problem 11	**WORKED EXAMPLE – Characteristics of Logarithmic Functions**

The Logarithmic Function in Problem 7 is of the form $f(x) = \log_b x, (b > 0 \text{ and } b \neq 1)$. All Logarithmic Functions of this form share key characteristics. In this example, we look at a typical graph of this type of function and list the key characteristics in the table below.

$$f(x) = \log_b x, (b > 0 \text{ and } b \neq 1)$$



Domain	$x > 0$ (all positive real numbers)
Range	All real numbers
Horizontal Intercept	(1, 0)
Vertical Asymptote	$x = 0$
Vertical Intercept	Does not exist
Left to Right Behavior	The function is always increasing but more and more slowly (at a decreasing rate)
Values of x for which $f(x) > 0$	$x > 1$
Values of x for which $f(x) < 0$	$0 < x < 1$
Values of x for which $f(x) = 0$	$x = 1$
Values of x for which $f(x) = 1$	$x = b$ because $\log_b b = 1$

Problem 12	**YOU TRY – GRAPHING LOGARITHMIC FUNCTIONS**

Graph $g(x) = \log_6 x$ on your graphing calculator. Use window x: [0..10] and y: [-2..2]. Use Change of Base to rewrite your function before graphing. Draw an accurate graph in the space below and fill in the table.

$g(x) = \log_6 x =$ ________________________

(rewrite using Change of Base)

a) Domain	
b) Range	
c) Horizontal Intercept	
d) Vertical Intercept	
e) Vertical Asymptote	
f) Values of x for which $g(x)$ is increasing	
g) Values of x for which $g(x) > 0$	
h) Values of x for which $g(x) < 0$	
i) Values of x for which $g(x) = 0$	
j) Values of x for which $g(x) = 1$	

Section 7.4 – Solving Logarithmic Equations

We will use what we now know about Logarithmic and Exponential forms to help us solve Logarithmic Equations. There is a step-by-step process to solve these types of equations. Try to learn this process and apply it to these types of problems.

Solving Logarithmic Equations – By Changing to Exponential Form

Solving logarithmic equations involves these steps:

1. **ISOLATE** the logarithmic part of the equation
2. Change the equation to **EXPONENTIAL** form
3. **ISOLATE** the variable
4. **CHECK** your result if possible
5. **IDENTIFY** the final result in **EXACT** form then in rounded form as indicated by the problem

Notes:

- To ISOLATE means to manipulate the equation using addition, subtraction, multiplication, and division so that the Log part and its input expression are by themselves.
- EXACT FORM for an answer means an answer that is not rounded until the last step

Problem 13	**MEDIA EXAMPLE – SOLVING LOGARITHMIC EQUATIONS**
Solve $\log_3 x = 2$ for x	Original Problem Statement
	Step 1: **ISOLATE** the logarithmic part of the equation
	Step 2: Change the equation to **EXPONENTIAL** form
	Step 3: **ISOLATE** the variable
	Step 4: **CHECK** your result if possible
	Step 5: **IDENTIFY** the final result in **EXACT** form then in rounded form as indicated by the problem

Problem 14	**WORKED EXAMPLE - SOLVING LOGARITHMIC EQUATIONS**
Solve $3 + \log_3(x - 1) = 7$ for x	Original Problem Statement
Subtract 3 from both sides $\log_3(x - 1) = 4$	Step 1: **ISOLATE** the logarithmic part of the equation
$3^4 = x - 1$ $81 = x - 1$ $\mathbf{82 = x}$	Step 2: Change the equation to **EXPONENTIAL** form and Step 3: **ISOLATE** the variable
$\log_3(82 - 1) = \log_3(81)$ $= 4$ because $3^4 = 81$ Therefore $3 + \log_3(82 - 1) = 7$ $3 + 4 = 7$ $7 = 7$ CHECKS	Step 4: **CHECK** your result if possible
$\boldsymbol{x} = \mathbf{82}$ (this is exact)	Step 5: **IDENTIFY** the final result in **EXACT** form then in rounded form as indicated by the problem

Problem 15	**MEDIA EXAMPLE - SOLVING LOGARITHMIC EQUATIONS**
Solve $4 + 6\log_2(3x + 2) = 5$ for x	Original Problem Statement
	Step 1: **ISOLATE** the logarithmic part of the equation
	Step 2: Change the equation to **EXPONENTIAL** form
	Step 3: **ISOLATE** the variable
	Step 4: **CHECK** your result if possible
	Step 5: **IDENTIFY** the final result in **EXACT** form then in rounded form as indicated by the problem

Problem 16	**YOU TRY - SOLVING LOGARITHMIC EQUATIONS**
Solve $\log_2(x-1)=5$ for x	Original Problem Statement
	Step 1: ISOLATE the logarithmic part of the equation
	Step 2: Change the equation to EXPONENTIAL form
	Step 3: ISOLATE the variable
	Step 4: CHECK your result if possible
	Step 5: IDENTIFY the final result in EXACT form then in rounded form as indicated by the problem

Problem 17	**YOU TRY - SOLVING LOGARITHMIC EQUATIONS**
Solve $5+4\log_3(7x+1)=8$ for x	Original Problem Statement
	Step 1: ISOLATE the logarithmic part of the equation
	Step 2: Change the equation to EXPONENTIAL form
	Step 3: ISOLATE the variable
	Step 4: CHECK your result if possible
	Step 5: IDENTIFY the final result in EXACT form then in rounded form as indicated by the problem

What's all the fuss? Exact form? Approximate Form? Why does it matter?

If you wanted to approximate the fraction $\frac{1}{3}$, what would you say? Probably that $\frac{1}{3}$ is about .3, right?

But what does $\frac{1}{3}$ ACTUALLY equal? Well, it equals .33333333333 repeating forever. Any number of decimals that we round to in order to represent $\frac{1}{3}$ is an APPROXIMATION.
The only EXACT representation of $\frac{1}{3}$ is $\frac{1}{3}$.

So what difference does this make? Suppose you wanted to compute $4^{1/3}$. Look at the following computations to as many decimals as we can.

$4^{1/3}$ = 4^(1/3) on your calculator = 1.587401052
$4^{.3}$ = 4^.3 on your calculator = 1.515716567

The final computation results are not the same but they are pretty close depending on where we would round the final resul. Which one is more accurate? The $4^{1/3}$ is more accurate because we used EXACT form for $\frac{1}{3}$.

What happens when the base of the exponential is much larger? Suppose you want to compute $1025^{1/3}$.

$1025^{1/3}$ = 1025^(1/3) = 10.08264838
$1025^{.3}$ = 1025^(.3) = 8.002342949

These two results are quite a bit different and this type of behavior only gets worse as the numbers you are working with get larger. So, remember, if you are asked to compute a result to EXACT form, do not round any computation in your solution process until the very end.

Section 7.5 – Solving Exponential Equations Algebraically and Graphically

We will use what we now know about Logarithmic and Exponential forms and Change of Base formula to help us solve Exponential Equations.

Problem 18	**MEDIA EXAMPLE – SOLVE EXPONENTIAL EQUATIONS**
Solve $3^x = 25$ for x Round the final result to three decimal places.	Original Problem Statement
	Step 1: ISOLATE the exponential part of the equation
	Step 2: Change the equation to LOGARITHMIC form
	Step 3: ISOLATE the variable
	Step 4: IDENTIFY the final result in EXACT form then in rounded form as indicated by the problem. You may need to use Change of Base here to compute your logarithm.
	Step 5: CHECK your result by using the graphing (intersection) method to solve the original problem

Problem 19	**MEDIA EXAMPLE – SOLVE EXPONENTIAL EQUATIONS**
Solve $11.36(1.080)^t = 180$ for t Round the final result to three decimal places.	Original Problem Statement
	Step 1: ISOLATE the exponential part of the equation
	Step 2: Change the equation to LOGARITHMIC form
	Step 3: ISOLATE the variable
	Step 4: IDENTIFY the final result in EXACT form then in rounded form as indicated by the problem. You may need to use Change of Base here to compute your logarithm.
	Step 5: CHECK your result by using the graphing (intersection) method to solve the original problem

Problem 20	**YOU TRY – SOLVE EXPONENTIAL EQUATIONS**
Solve for $12.5 + 3^x = 17.8$ Round the final result to three decimal places.	Original Problem Statement
	Step 1: ISOLATE the exponential part of the equation
	Step 2: Change the equation to LOGARITHMIC form
	Step 3: ISOLATE the variable
	Step 4: IDENTIFY the final result in EXACT form then in rounded form as indicated by the problem. You may need to use Change of Base here to compute your logarithm.
	Step 5: CHECK your result by using the graphing (intersection) method to solve the original problem. Sketch the graph at in the space at left.

Section 7.6 – Using Logarithms as a Scaling Tool

Logarithms are used in the sciences particularly in biology, astronomy and physics. The Richter scale measurement for earthquakes is based upon logarithms, and logarithms formed the foundation of our early computation tool (pre-calculators) called a Slide Rule.

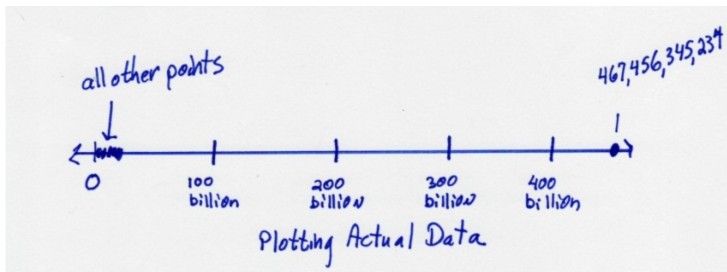
One of the unique properties of Logarithms is their ability to scale numbers of great or small size so that these numbers can be understood and compared. Let's see how this works with an example.

Problem 21	WORKED EXAMPLE – USING LOGARITHMS AS A SCALING TOOL

Suppose you are given the following list of numbers and you want to plot them all on the same number line:

Plot 0.00000456, 0.00372, 1.673, 1356, 123,045 and 467,456,345,234.

If we scale to the larger numbers, then the smaller numbers blend together and we can't differentiate them.



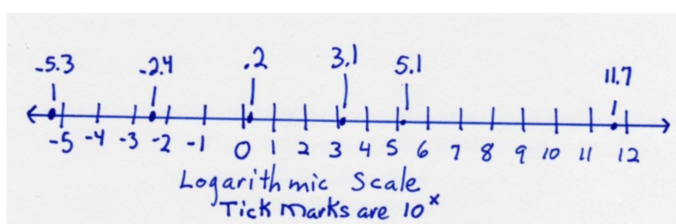
Let's use logarithms and create a logarithmic scale and see how that works. First, make a table that translates your list of numbers into logarithmic form by taking the "log base 10" or common logarithm of each value.

Original #	0.00000456	0.00372	1.673	1356	123,045	467,456,345,234
Log (#)	-5.3	-2.4	.2	3.1	5.1	11.7

Then, redraw your number line and plot the logarithmic value for each number.

Notice that labeling your scale as a logarithmic scale is VERY important. Otherwise, you may not remember to translate back to the actual data and you may forget that your tick marks are not unit distances.

The new scale gives you an idea of the relative distance between your numbers and allows you to plot all your numbers at the same time. To understand the distance between each tick mark, remember that the tick mark label is the exponent on 10 (base of the logarithm used). So from 1 to 2 is a distance of $10^2 - 10^1 = 100 - 10 = 90$. The distance between 2 and 3 is $10^3 – 10^2$ or $1000 – 100 = 900$, etc…



You will learn a LOT more about logarithmic scaling if you take science classes, as this is just a very brief introduction to the idea.