Lesson 5 – Introduction to Exponential Functions

Exponential Functions play a major role in our lives. Many of the challenges we face involve exponential change and can be modeled by an Exponential Function. Financial considerations are the most obvious, such as the growth of our retirement savings, how much interest we are paying on our home loan or the effects of inflation.

In this lesson, we begin our investigation of Exponential Functions by comparing them to Linear Functions, examining how they are constructed and how they behave. We then learn methods for solving exponential functions given the input and given the output.

Lesson Topics:

Section 5.1: Linear Functions Vs. Exponential Functions

- Characteristics of linear functions
- Comparing linear and exponential growth
- Using the common ratio to identify exponential data
- Horizontal Intercepts

Section 5.2: Characteristics of Exponential Functions

Section 5.3: Solving Exponential Equations by Graphing

- Using the Intersect Method to solve exponential equations on the graphing calculator
- Guidelines for setting an appropriate viewing window

Section 5.4: Applications of Exponential Functions

Lesson 5 Checklist

Component	Required? Y or N	Comments	Due	Score
Mini-Lesson				
Online Homework				
Online Quiz				
Online Test				
Practice Problems				
Lesson Assessment				

Name: ______________________________ **Date:** ________________

Mini-Lesson 5

Section 5.1 – Linear Functions vs. Exponential Functions

Problem 1	YOU TRY – Characteristics of Linear Functions

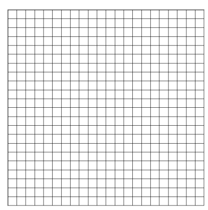
Given a function, $f(x) = mx + b$, respond to each of the following. Refer back to previous lessons as needed.

a) The variable x represents the __________________ quantity.

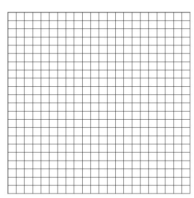
b) $f(x)$ represents the _________________ quantity.

c) The graph of f is a _________________________________ with slope __________ and vertical intercept _________________ .

d) On the graphing grid below, draw an INCREASING linear function. In this case, what can you say about the slope of the line? m ______ 0 (Your choices here are > or <)



e) On the graphing grid below, draw a DECREASING linear function. In this case, what can you say about the slope of the line? m ______ 0 (Your choices here are > or <)



f) The defining characteristic of a LINEAR FUNCTION is that the RATE OF CHANGE (also called the SLOPE) is _______________________.

g) The domain of a LINEAR FUNCTION is _____________________________________

This next example is long but will illustrate the key difference between EXPONENTIAL FUNCTIONS and LINEAR FUNCTIONS.

Problem 2	**WORKED EXAMPLE – DOLLARS & SENSE**

On December 31st around 10 pm, you are sitting quietly in your house watching Dick Clark's New Year's Eve special when there is a knock at the door. Wondering who could possibly be visiting at this hour you head to the front door to find out who it is. Seeing a man dressed in a three-piece suit and tie and holding a briefcase, you cautiously open the door.

The man introduces himself as a lawyer representing the estate of your recently deceased great uncle. Turns out your uncle left you some money in his will, but you have to make a decision. The man in the suit explains that you have three options for how to receive your allotment.

<u>Option A:</u> \$1000 would be deposited on Dec 31st in a bank account bearing your name and each day an additional \$1000 would be deposited (until January 31st).

<u>Option B:</u> One penny would be deposited on Dec 31st in a bank account bearing your name. Each day, the amount would be doubled (until January 31st).

<u>Option C:</u> Take \$30,000 on the spot and be done with it.

Given that you had been to a party earlier that night and your head was a little fuzzy, you wanted some time to think about it. The man agreed to give you until 11:50 pm. Which option would give you the most money after the 31 days???

A table of values for option A and B are provided on the following page. Before you look at the values, though, which option would you select according to your intuition?

Without "doing the math" first, I would instinctively choose the following option (circle your choice):

Option A	Option B	Option C

Option A: \$1000 to start + \$1000 per day		Option B: \$.01 to start then double each day	
Note that t = 0 on Dec. 31st			
Table of input/output values		Table of input/output values	
t = time in # of days since Dec 31	*A(t)*=\$ in account after *t* days	*t* = time in # of days since Dec 31	*B(t)* = \$ in account after *t* days
0	1000	0	.01
1	2000	1	.02
2	3000	2	.04
3	4000	3	.08
4	5000	4	.16
5	6000	5	.32
6	7000	6	.64
7	8000	7	1.28
8	9000	8	2.56
9	10,000	9	5.12
10	11,000	10	10.24
11	12,000	11	20.48
12	13,000	12	40.96
13	14,000	13	81.92
14	15,000	14	163.84
15	16,000	15	327.68
16	17,000	16	655.36
17	18,000	17	1,310.72
18	19,000	18	2,621.44
19	20,000	19	5,242.88
20	21,000	20	10,485.76
21	22,000	21	20,971.52
22	23,000	22	41,943.04
23	24,000	23	83,886.08
24	25,000	24	167,772.16
25	26,000	25	335,544.32
26	27,000	26	671,088.64
27	28,000	27	1,342,177.28
28	29,000	28	2,684,354.56
29	30,000	29	5,368,709.12
30	31,000	30	10,737,418.24
31	32,000	31	21,474,836.48

WOWWWWW!!!!!!!

What IS that number for Option B? I hope you made that choice… it's 21 million, 4 hundred 74 thousand, 8 hundred 36 dollars and 48 cents. Let's see if we can understand what is going on with these different options.

Problem 3	**MEDIA EXAMPLE – Compare Linear and Exponential Growth**

For the example discussed in Problem 2, respond to the following:

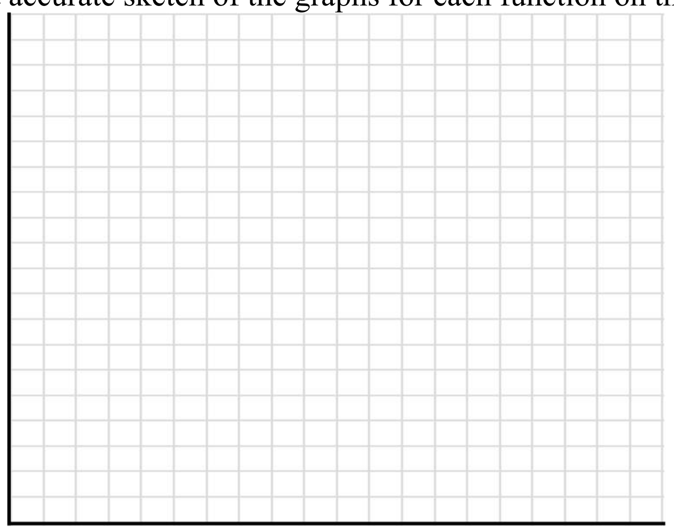
a) Symbolic representation (model) for each situation:

A(t) = Type of function ___________________

B(t) = Type of function ___________________

C(t) = Type of function ___________________

b) Provide a rough but accurate sketch of the graphs for each function on the same grid below:



c) What are the practical domain and range for each function?

	Practical Domain	Practical Range
A(t):		
B(t):		
C(t):		

d) Based on the graphs, which option would give you the most money after 31 days?

e) Let's see if we can understand WHY option B grows so much faster. Let's focus just on options A and B. Take a look at the data tables given for each function. Just the later parts of the initial table are provided.

$A(t) = 1000t + 1000$

t = time in # of days since Dec 31	$A(t)$=\$ in account after t days
20	21,000
21	22,000
22	23,000
23	24,000
24	25,000
25	26,000
26	27,000
27	28,000
28	29,000
29	30,000
30	31,000
31	32,000

$B(t) = .01(2)^t$

t =time in # of days since Dec 31	$B(t)$ = \$ in account after t days
20	10,485.76
21	20,971.52
22	41,943.04
23	83,886.08
24	167,772.16
25	335,544.32
26	671,088.64
27	1,342,177.28
28	2,684,354.56
29	5,368,709.12
30	10,737,418.24
31	21,474,836.48

As t increases from day 20 to 21, describe how the outputs change for each function:

$A(t)$:

$B(t)$:

As t increases from day 23 to 24, describe how the outputs change for each function:

$A(t)$:

$B(t)$:

So, in general, we can say as the inputs increase from one day to the next, then the outputs for each function:

$A(t)$:

$B(t)$:

In other words, $A(t)$ grows ______________________ and $B(t)$ grows ______________________.

We have just identified the primary difference between **LINEAR FUNCTIONS** and **EXPONENTIAL FUNCTIONS.**

Exponential Functions vs. Linear Functions

The outputs for **Linear Functions** change by **ADDITION** and the outputs for **Exponential Functions** change by **MULTIPLICATION**.

Problem 4	**WORKED EXAMPLE – Are the Data Exponential?**

To determine if an exponential function is the best model for a given data set, calculate the ratio $\frac{y_2}{y_1}$ for each of the consecutive points. If this ratio is approximately the same for the entire set, then an exponential function models the data best. For example:

x	1	2	3	4	5
y	1.75	7	28	112	448

For this set of data, $\frac{y_2}{y_1} = \frac{7}{1.75} = \frac{28}{7} = \frac{112}{28} = \frac{448}{112} = 4$

Since $\frac{y_2}{y_1} = 4$ for all consecutive pairs, the data are exponential with a growth factor of 4.

Problem 5	**MEDIA EXAMPLE – Linear Data Vs. Exponential Data**

Analyze each of the following data sets to determine if the set can be modeled best by a linear function or an exponential function. Write the equation that goes with each data set. *[Note that in the video for the first table, the first two x-values are incorrectly reversed.]*

x	-2	-1	0	1	2	3	4
y	$\frac{1}{25}$	$\frac{1}{5}$	1	5	25	125	625

x	0	1	2	3	4	5	6	7
y	-3.2	-3	-2.8	-2.6	-2.4	-2.2	-2.0	-1.8

Problem 6	**YOU TRY – Use Common Ratio to Identify Exponential Data**

a) Given the following table, explain why the data can be best modeled by an exponential function. Use the idea of common ratio in your response.

x	0	1	2	3	4	5	6
$f(x)$	15	12	9.6	7.68	6.14	4.92	3.93

b) Determine an exponential model $f(x) = ab^x$ that fits these data. Start by identifying the values of a and b and then write your final result using proper notation.

c) Determine $f(10)$. Round to the nearest hundredth.

d) Determine $f(50)$. Write your answer as a decimal *and* in scientific notation.

Section 5.2 – Characteristics of Exponential Functions

Exponential Functions are of the form $f(x) = ab^x$

where a = the INITIAL VALUE

b = the base ($b > 0$ and $b \neq 1$); also called the GROWTH or DECAY FACTOR

Important Characteristics of the EXPONENTIAL FUNCTION $f(x) = ab^x$

- x represents the INPUT quantity
- $f(x)$ represents the OUTPUT quantity
- The graph of $f(x)$ is in the shape of the letter "J" with vertical intercept $(0, a)$ and base b (note that b is the same as the COMMON RATIO from previous examples)
- If $b > 1$, the function is an EXPONENTIAL GROWTH function, and the graph INCREASES from left to right
- If $0 < b < 1$, the function is an EXPONENTIAL DECAY function, and the graph DECREASES from left to right
- Another way to identify the vertical intercept is to evaluate $f(0)$.

Problem 7	WORKED EXAMPLE – Examples of Exponential Functions

a) $f(x) = 2(3)^x$

Initial Value, $a = 2$, Vertical Intercept = $(0, 2)$
Base, $b = 3$.
$f(x)$ is an exponential GROWTH function since $b > 1$.

b) $g(x) = 1523(1.05)^x$

Initial Value, $a = 1523$, Vertical Intercept = $(0, 1523)$
Base, $b = 1.05$.
$g(x)$ is an exponential GROWTH function since $b > 1$.

c) $h(x) = 256(0.85)^x$

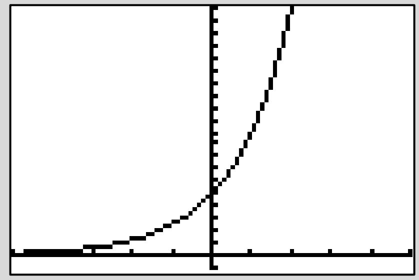
Initial Value, $a = 256$, Vertical Intercept = $(0, 256)$
Base, $b = 0.85$.
$h(x)$ is an exponential DECAY function since $b < 1$.

d) $k(x) = 32(0.956)^x$

Initial Value, $a = 32$, Vertical Intercept = $(0, 32)$
Base, $b = 0.956$.
$k(x)$ is an exponential DECAY function since $b < 1$.

Graph of a generic Exponential Growth Function

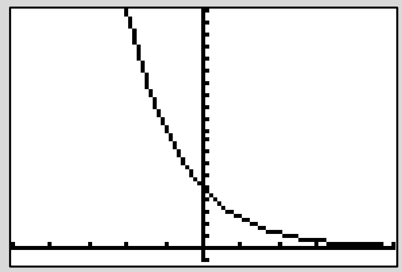
$f(x) = ab^x$, $b > 1$



- Domain: All Real Numbers
- Range: $f(x) > 0$
- Horizontal Intercept: None
- Vertical Intercept: $(0, a)$
- Horizontal Asymptote: $y = 0$
- Left to right behavior of the function: INCREASING

Graph of a generic Exponential Decay Function

$f(x) = ab^x$, $0 < b < 1$



- Domain: All Real Numbers
- Range: $f(x) > 0$
- Horizontal Intercept: None
- Vertical Intercept: $(0, a)$
- Horizontal Asymptote: $y = 0$
- Left to right behavior of the function: DECREASING

Problem 8	MEDIA EXAMPLE – Characteristics of Exponential Functions

Consider the function $f(x) = 12(1.45)^x$

Initial Value (a): ____________

Base (b):_____________

Domain:______________________________________

Range:______________________________________

Horizontal Intercept: ______________________

Vertical Intercept:_______________________

Horizontal Asymptote: ________________________

Increasing or Decreasing? ______________________

Problem 9	**YOU TRY – Characteristics of Exponential Functions**

Complete the table. Start by graphing each function using the indicated viewing window. Sketch what you see on your calculator screen.

	$f(x) = 335(1.25)^x$	$g(x) = 120(0.75)^x$
Graph Use Viewing Window: Xmin = -10 Xmax = 10 Ymin = 0 Ymax = 1000		
Initial Value (a)?		
Base (b)?		
Domain? (Use Inequality Notation)		
Range? (Use Inequality Notation)		
Horizontal Intercept?		
Vertical Intercept?		
Horizontal Asymptote? (Write the *equation*)		
Increasing or Decreasing?		

Section 5.3 – Solving Exponential Equations by Graphing

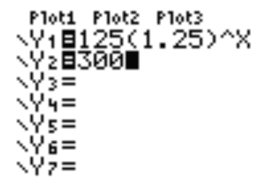
Problem 10	**WORKED EXAMPLE – Solving Exponential Equations by Graphing**

Solve the equation $125(1.25)^x = 300$. Round your answer to two decimal places.

To do this, we will use a process called the INTERSECTION METHOD on our graphing calculators.

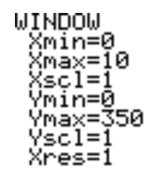
To solve $125(1.25)^x = 300$

- Press Y= then enter Y1 = $125(1.25)^x$ and Y2 = 300
  Note: You could also let Y1 = 300 and Y2=$125(1.25)^x$



- Press WINDOW then enter the values at right.

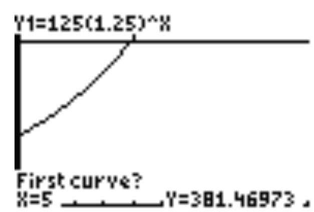
Try to determine why these values were selected. You must see the intersection in your window. Other entries will work. If you graph and do not see both graphs AND where they intersect, you must pick new WINDOW values until you do.



- Press 2^{nd}>CALC
- Scroll to 5: INTERSECT and press ENTER

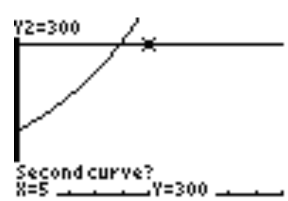
Notice the question, "First Curve?" The calculator is asking if Y1 = $125(1.25)^x$ is the first curve in the intersection.

- Press Enter to indicate "Yes"

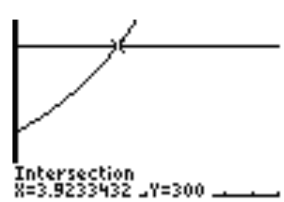


Notice the question, "Second Curve?" The calculator is asking if Y2 = 300 is the second curve in the intersection.

- Press Enter to indicate "Yes"



- Press Enter at the "Guess" question and obtain the screen at right. Your intersection values are given at screen bottom and the intersection is marked with a cursor. Round as indicated in your problem directions or as dictated by the situation.



Our answer is x = 3.92. Note that this information corresponds to the ordered pair (3.92, 300).

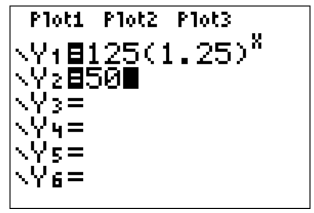
Problem 11	**WORKED EXAMPLE – Solving Exponential Equations by Graphing**

Given $f(x) = 125(1.25)^x$ find x when $f(x) = 50$. Round your respond to two decimal places.

To do this, we need to SOLVE the equation $125(1.25)^x = 50$ using the INTERSECTION METHOD.

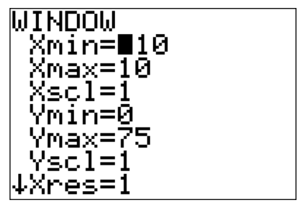
To solve $125(1.25)^x = 50$

- Press Y= then enter Y1 = $125(1.25)^x$ and Y2 = 50
  Note: You could also let Y1 = 50 and Y2=$125(1.25)^x$

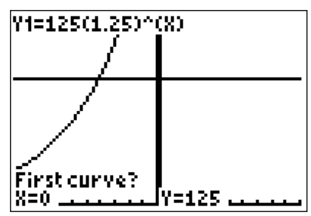


- Press WINDOW then enter the values at right.

Try to determine why these values were selected. You must see the intersection in your window. Other entries will work. If you graph and do not see both graphs AND where they intersect, you must pick new WINDOW values until you do.

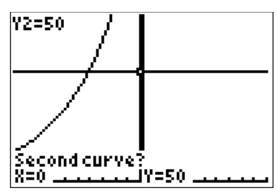


- Press 2nd>CALC
- Scroll to 5: INTERSECT and press ENTER



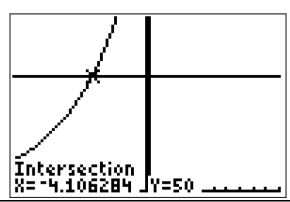
Notice the question, "First Curve?" The calculator is asking if Y1 = $125(1.25)^x$ is the first curve in the intersection.

- Press Enter to indicate "Yes"



Notice the question, "Second Curve?" The calculator is asking if Y2 = 50 is the second curve in the intersection.

- Press Enter to indicate "Yes"



- Press Enter at the "Guess" question and obtain the screen at right. Your intersection values are given at screen bottom and the intersection is marked with a cursor. Round as indicated in your problem directions or as dictated by the situation.

For this problem, we were asked to find x when $f(x) = 50$. Round to two decimal places. Our response is that, "When $f(x) = 50$, $x = -4.11$". Note that this information corresponds to the ordered pair (-4.11, 50) on the graph of $f(x) = 125(1.25)^x$

GUIDELINES FOR SELECTING WINDOW VALUES FOR INTERSECTIONS

While the steps for using the INTERSECTION method are straightforward, choosing values for your window are not always easy. Here are some guidelines for choosing the edges of your window:

- First and foremost, the intersection of the equations MUST appear clearly in the window you select. Try to avoid intersections that appear just on the window's edges, as these are hard to see and your calculator will often not process them correctly.
- Second, you want to be sure that other important parts of the graphs appear (i.e. where the graph or graphs cross the y-axis or the x-axis).
- When choosing values for x, start with the standard XMin = -10 and Xmax = 10 UNLESS the problem is a real-world problem. In that case, start with Xmin=0 as negative values for a world problem are usually not important. If the values for Xmax need to be increased, choose 25, then 50, then 100 until the intersection of graphs is visible.
- When choosing values for y, start with Ymin = 0 unless negative values of Y are needed for some reason. For Ymax, all graphs need to appear on the screen. So, if solving something like $234(1.23)^x = 1000$, then choose Ymax to be bigger than 1000 (say, 1500).

If the intersection does not appear in the window, then try to change only one window setting at a time so you can clearly identify the effect of that change (i.e. make Xmax bigger OR change Ymax but not both at once). Try to think about the functions you are working with and what they look like and use a systematic approach to making changes.

Problem 12	**MEDIA EXAMPLE – Solving Exponential Equations by Graphing**

Solve the equation $400 = 95(0.89)^x$. Round your answer to two decimal places.

Problem 13	**YOU TRY – Window Values and Intersections**

In each situation below, you will need to graph to find the solution to the equation using the INTERSECTION method described in this lesson. Fill in the missing information for each situation. Include a rough but accurate sketch of the graphs and intersection point. Mark and label the intersection. Round answers to two decimal places.

a) Solve $54(1.05)^x = 250$ Solution: x = ______________

Xmin:__________

Xmax:__________

Ymin:__________

Ymax:__________

b) Solve $2340(0.82)^x = 1250$ Solution: x = ______________

Xmin:__________

Xmax:__________

Ymin:__________

Ymax:__________

c) Solve $45 = 250(1.045)^x$ Solution: x = ______________

Xmin:__________

Xmax:__________

Ymin:__________

Ymax:__________

Section 5.4 – Applications of Exponential Functions

Writing Exponential Equations/Functions

Given a set of data that can be modeled using an exponential equation, use the steps below to determine the particulars of the equation:

1. Identify the initial value. This is the *a* part of the exponential equation $y = ab^x$. To find *a*, look for the starting value of the data set (the output that goes with input 0).
2. Identify the common ratio, *b*, value. To do this, make a fraction of two consecutive outputs (as long as the inputs are separated by exactly 1). We write this as the fraction $\frac{y_2}{y_1}$ to indicate that we put the second *y* on top and the first on the bottom. Simplify this fraction and round as the problem indicates to obtain the value of *b*.
3. Plug in the values of *a* and *b* into $y = ab^x$ to write the exponential equation.
4. Replace *y* with appropriate notation as needed to identify a requested exponential FUNCTION.

Problem 14	**MEDIA EXAMPLE – Writing Exponential Equations/Functions**

The population of a small city is shown in the following table.

Year	Population
2000	12,545
2001	15,269
2002	18,584

Assume that the growth is exponential. Let $t = 0$ represent the year 2000. Let *a* be the initial population in 2000. Let *b* equal the ratio in population between the years 2000 and 2001.

a) Write the equation of the exponential model for this situation. Round any decimals to two places. Be sure your final result uses proper function notation.

b) Using this model, forecast the population in 2008 (to the nearest person).

c) Also using this model, determine the nearest whole year in which the population will reach 50,000.

Problem 15	**YOU TRY – Writing Exponential Equations/Functions**

You have just purchased a new car. The table below shows the value, V, of the car after *n* years.

n = number of years	V = Value of Car
0	24,800
1	21,328
2	18,342

a) Assume that the depreciation is exponential. Write the equation of the exponential model for this situation. Round any decimals to two places. Be sure your final result uses proper function notation.

b) You finance the car for 60 months. What will the value of the car be when the loan is paid off? Show all steps. Write your answer in a complete sentence.

Problem 16	**YOU TRY – Writing Exponential Equations/Functions**

In 2010, the population of Gilbert, AZ was about 208,000. By 2011, the population had grown to about 215,000.

a) Assuming that the growth is exponential, construct an exponential model that expresses the population, P, of Gilbert, AZ x years since 2010. Your answer must be written in function notation. Round to three decimals as needed.

b) Use this model to predict the population of Gilbert, AZ in 2014. Write your answer in a complete sentence.

c) According to this model, *in what year* will the population of Gilbert, AZ reach 300,000? (Round your answer DOWN to the nearest whole year.)

Problem 17	**YOU TRY – Applications of Exponential Functions**

One 8-oz cup of coffee contains about 100 mg of caffeine. The function $A(x) = 100(0.88)^x$ gives the amount of caffeine (in mg) remaining in the body x hours after drinking a cup of coffee. Answer in complete sentences.

a) Identify the vertical intercept of this function. Write it as an ordered pair and interpret its meaning in a complete sentence.

b) How much caffeine remains in the body 8 hours after drinking a cup of coffee? Round your answer to two decimal places as needed.

c) How long will it take the body to metabolize half of the caffeine from one cup of coffee? (i.e. How long until only 50mg of caffeine remain in the body?) Show all of your work, and write your answer in a complete sentence. Round your answer to two decimal places as needed.

d) According to this model, how long will it take for *all* of the caffeine to leave the body?