Lesson 11 –Rational Functions

In this lesson, you will embark on a study of rational functions. These may be unlike any function you have ever seen. Rational functions look different because they are in pieces but understand that the image presented is that of a single function.

In this lesson, you will graph rational functions and solve rational equations both graphically and algebraically. You will finish the lesson with an application of rational functions.

Lesson Topics

Section 11.1: Characteristics of Rational Functions

- Domain
- Vertical Asymptotes
- Horizontal Asymptotes

Section 11.2: Solving Rational Equations

- Solve by graphing
- Solve algebraically
- Determine Horizontal and Vertical Intercepts
- Working with Input and Output

Section 11.3: Applications of Rational Functions

Lesson 11 Checklist

Component	Required? Y or N	Comments	Due	Score
Mini-Lesson				
Online Homework				
Online Quiz				
Online Test				
Practice Problems				
Lesson Assessment				

Name: ____________________ **Date:** ______________

Mini-Lesson 11

Section 11.1 – Characteristics of Rational Functions

A RATIONAL FUNCTION is a function of the form

$$f(x) = \frac{p(x)}{q(x)}$$

where $p(x)$ and $q(x)$ are polynomials and $q(x)$ does not equal zero (remember that division by zero is undefined). Rational functions have similar shapes depending on the degree of the polynomials $p(x)$ and $q(x)$. However, the shapes are different enough that only the general characteristics are listed here and not a general graph:

DOMAIN

- The solution(s) found when solving $q(x) = 0$ are the values that are NOT part of the domain of $f(x)$.

VERTICAL ASYMPTOTES

- $f(x)$ will have VERTICAL ASYMPTOTES at all input values where $q(x) = 0$. These asymptotes are vertical guiding lines for the graph of $f(x)$
- The graph of $f(x)$ will never cross over these lines.
- To find the Vertical Asymptotes for $f(x)$, set the denominator, $q(x)$, equal to 0 and solve for x. For each solution, x = that value is the equation of your vertical asymptote.

HORIZONTAL ASYMPTOTES

This asymptote is a guiding line for the function as the input values approach positive and negative infinity. If $f(x)$ has a HORIZONTAL ASYMPTOTE $y = a$, then as the input values approach positive and negative infinity, the output values will approach a.

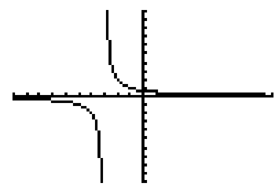
Two methods for finding the Horizontal Asymptote of a Rational Function:

1. Enter $f(x)$ into your graphing calculator, then go to your TABLE, and enter in "large" values for the input. Start with 10, then 100, 1000, and so on. If a horizontal asymptote exists, you will notice the output values "approaching" a number, a.
   The line $y = a$ is the horizontal asymptote.
2. Make a fraction with only the highest degree term in $p(x)$ as the numerator and the highest degree term in $q(x)$ as the denominator. Reduce this fraction completely.
   - If the fraction reduces to a number, then y = that number is the equation of the horizontal asymptote.
   - If the fraction reduces to $\frac{\text{number}}{x}$, then $y = 0$ is your horizontal asymptote
   - $f(x)$ will have a horizontal asymptote only if the degree of $q(x) \geq$ degree of $p(x)$.

Problem 1	**WORKED EXAMPLE – Key Characteristics of Rational Functions**

Graph $f(x)=\dfrac{2}{x+3}$ and determine the horizontal and vertical asymptotes and the domain.

To graph, let Y1 = $\dfrac{2}{x+3}$. Input Y1=2/(x+3) into your Y= list and note the use of ( ).



To find any VERTICAL ASYMPTOTES, set the denominator equal to 0 and solve for x.
$x + 3 = 0$, therefore $x = –3$. The equation of the vertical asymptote is $x = –3$.

To find the DOMAIN, set the denominator equal to 0 and solve for x. Because $x = –3$ makes the denominator of $f(x)$ equal zero, this value is not part of the domain. All other inputs are allowable. So, the domain for f(x) is $x \neq –3$, "all real numbers except –3".

To find the HORIZONTAL ASYMPTOTE, make a fraction of the highest power term in the numerator (2) and the highest power term in the denominator (x). Reduce. Here is what the fraction looks like.

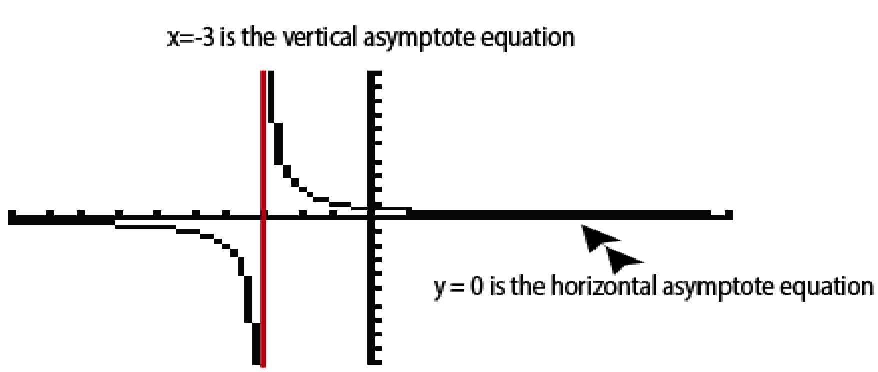
$$\frac{2}{x}$$

What you are trying to find out is, what is the value of this function as x gets really big (positive) and really big (negative)? To answer this question, we need to apply a little abstract thinking.

ABSTRACT THINKING

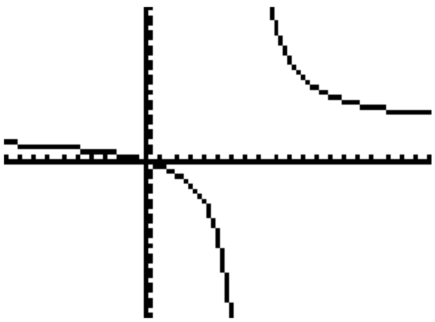
- In your mind, think of the very biggest positive number you can think of. What happens when you divide 2 by that number? Well, the result is going to be very, very small…effectively zero if your number is big enough. So, y = 0 is your horizontal asymptote equation as the same thing works for the biggest negative number you can think of.

Putting all these things together gives the following graph with asymptotes labeled:

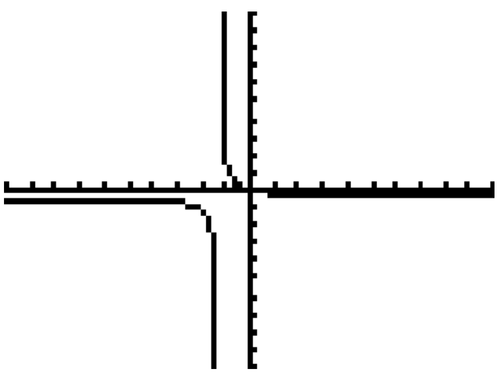


Problem 2	**MEDIA EXAMPLE – Key Characteristics of Rational Functions**

a) Graph $f(x) = \dfrac{4x}{x-7}$ and determine the horizontal and vertical asymptotes and the domain.



b) Graph $f(x) = \dfrac{-3x}{7x+9}$ and determine the horizontal and vertical asymptotes and the domain.



c) **Quick Example:** Find the Horizontal Asymptote for $f(x) = \dfrac{x-1}{x+5}$.

The ratio of highest degree terms in the numerator/denominator is y = $\dfrac{x}{x} = 1$
so the Horizontal Asymptote for this function is $y = 1$.

Problem 3	**YOU TRY – Key Characteristics of Rational Functions**

a) Graph $f(x) = \dfrac{4}{x-5}$ and determine the horizontal and vertical asymptotes and the domain.

Domain:

Vertical Asymptote:

Horizontal Asymptote:

b) Graph $g(x) = \dfrac{3x}{2x+1}$ and determine the horizontal and vertical asymptotes and the domain.

Domain:

Vertical Asymptote:

Horizontal Asymptote:

c) Graph $h(x) = \dfrac{2x+1}{4-x}$ and determine the horizontal and vertical asymptotes and the domain.

Domain:

Vertical Asymptote:

Horizontal Asymptote:

Section 11.2 – Solving Rational Equations

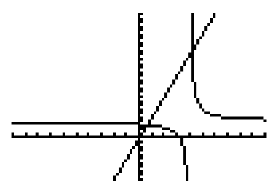
To solve Rational Equations by GRAPHING:

- Let Y1 = one side of the equation
- Let Y2 = other side of the equation
- Determine an appropriate window to see important parts of the graph
- Use the intersection method
- You may have more than one solution
- The *x*-value(s) of the intersection are your solutions

Problem 4	**WORKED EXAMPLE – Solve Rational Equations by Graphing**

Solve $5x = 4 + \dfrac{3}{x-4}$

Let Y1 = 5*x*
Let Y2 = 4 + 3/(*x* – 4) Note use of ( )
Graph on window x: [–10..10], y:[–10..30]
If you use standard window you do not see the upper intersection.



You will need to perform the intersection process two separate times on your calculator. One time, you should get x = 0.62 (the left intersection) and the second time you should get x = 4.18. Be sure to move your cursor far enough (it has to go all the way across the vertical asymptote) to read the second intersection. Solutions, then, are *x* = 0.62, 4.18

Problem 5	**MEDIA EXAMPLE – Solve Rational Equations by Graphing**

Solve $3 = 1 + \dfrac{3x}{x-1}$ by graphing. Round answer(s) to two decimals as needed.

To solve rational equations ALGEBRAICALLY (also called symbolically):

- Identify the common denominator for all fractions in the equation.
- Take note of the values of x that make the common denominator zero. These x-values cannot be used as solutions to the equation since we cannot divide by 0.
- Clear the fractions by multiplying both sides of the equation by the common denominator
- Solve for x.
- Check your work by plugging the value(s) back into the equation or by graphing.

Problem 6	WORKED EXAMPLE – Solve Rational Equations Algebraically

Solve $5x = 4 + \frac{3}{x-4}$ algebraically. Round solutions to two decimal places.

- Common denominator for all sides is $x – 4$. Multiply both sides of the equation by $(x – 4)$ and solve for x to get the following:

$$(5x)(x-4) = (4 + \frac{3}{x-4})(x-4)$$
$$5x^2 - 20x = 4(x-4) + \frac{3}{x-4}(x-4)$$
$$5x^2 - 20x = 4x - 16 + 3$$
$$5x^2 - 20x = 4x - 13$$
$$5x^2 - 20x - 4x + 13 = 0$$
$$5x^2 - 24x + 13 = 0$$

Notice that we now have a quadratic equation, which can be solved using the methods of last chapter. Because we are asked to solve our original problem algebraically, let's continue that process and not resort to a graphical solution. We will use the Quadratic Formula with $a = 5$, $b = –24$, and $c = 13$ to get:

$$x = \frac{-(-24) \pm \sqrt{(-24)^2 - 4(5)(13)}}{2(5)} = \frac{24 \pm \sqrt{576-260}}{10} = \frac{24 \pm \sqrt{316}}{10}$$

Because we want rounded solutions, I do NOT need to continue reducing my fraction solutions above but can compute the following directly:

$$x = \frac{24+\sqrt{316}}{10} \approx 4.18, \qquad x = \frac{24-\sqrt{316}}{10} = .62$$

These solutions match what we found in the graphing example previously.

To check, plug the values back into the original equation (one at a time) or use the graphing method.

Problem 7	**MEDIA EXAMPLE – Solving Rational Equations Algebraically**

Solve $3 = 1 + \dfrac{3x}{x-1}$ algebraically. Round answer(s) to two decimals as needed.

Problem 8	**YOU TRY – Solving Rational Equations Graphically/Algebraically**

Round answer(s) to two decimals as needed.

a) Solve $1 = \dfrac{5}{x-2} - 3$ graphically. Sketch the graph from your calculator screen, and indicate the viewing window you used.

Xmin=______

Xmax=______

Ymin=______

Ymax=______

Solution: _________________

b) Solve $1 = \dfrac{5}{x-2} - 3$ algebraically. Show complete work.

Problem 9	**MEDIA EXAMPLE – Working with Rational Functions**

Consider the function $f(x) = \dfrac{x-1}{x+5}$

a) What is the domain?

b) Give the ***equation*** of the vertical asymptote for $f(x)$.

c) Give the ***equation*** of the horizontal asymptote for $f(x)$.

d) What is the vertical intercept? Show your work.

e) What is the horizontal intercept? Show your work.

f) Determine $f(12)$. Show your work.

g) For what value of x is $f(x) = 3$? Show your work.

Problem 10	**YOU TRY – Working with Rational Functions**

Consider the function $g(x) = \dfrac{15x - 12}{3x + 4}$

a) What is the domain?

b) Give the ***equation*** of the vertical asymptote for $g(x)$.

c) Give the ***equation*** of the horizontal asymptote for $g(x)$.

d) What is the vertical intercept? Show your work.

e) What is the horizontal intercept? Show your work.

f) Determine $g(5)$. Show your work.

g) For what value of x is $g(x) = -8$? Show your work.

Section 11.3 – Applications of Rational Functions

Problem 11	**MEDIA EXAMPLE – Applications of Rational Functions**

You and your family are heading out to San Diego on a road trip. From Phoenix, the trip is 354.5 miles according to Google. Answer the following questions based upon this situation.

a) Use the relationship, Distance = Rate times Time or $d = rT$, to write a rational function $T(r)$ that has the rate of travel, r (in mph), as its input and the time of travel (in hours) as its output. The distance will be constant at 354.5 miles.

b) Provide a rough but accurate sketch of the graph in the space below. Label your horizontal and vertical axes. You only need to graph the first quadrant information. Indicate the graphing window you chose.

Xmin=______

Xmax=______

Ymin=______

Ymax=______

c) If you average 60 mph, how long will the trip take?

d) If the trip took 10 hours, what was your average rate of travel?

e) What does the graph indicate will happen as your rate increases?

f) What does the graph indicate will happen as your rate gets close to zero?

Problem 12	**YOU TRY – Applications of Rational Functions**

You and your friends are heading out to Las Vegas on a road trip. From Scottsdale, the trip is 308.6 miles according to Google. Answer the following questions based upon this situation.

a) Use the relationship, Distance = Rate times Time or $d = rT$, to write a rational function $T(r)$ that has the average rate of travel, r (in mph), as its input and the time of travel (in hours) as its output.

b) Provide a rough but accurate sketch of the graph in the space below. Label your horizontal and vertical axes. You only need to graph the first quadrant information. Indicate the graphing window you chose.

Xmin=______

Xmax=______

Ymin=______

Ymax=______

c) According to Google, the trip should take 5 hours. Determine your average rate of travel if the trip takes 5 hours.

d) Determine the vertical asymptote for $T(r)$, and write a sentence explaining its significance in this situation.

e) Determine the horizontal asymptote for $T(r)$, and write a sentence explaining its significance in this situation.